Noether's Theorem

Elsen Tjhung
School of Mathematics and Statistics, The Open University,
Walton Hall, Milton Keynes MK7 6AA, United Kingdom

A short note about Noether's theorem for translational symmetry in a Hamiltonian system.

I. TRANSLATIONAL SYMMETRY IN HAMILTONIAN

Let us consider some Hamiltonian:

$$H[\phi(\mathbf{r})] = \int \psi(\phi, \partial_\alpha \phi)\, d^3r, \tag{1}$$

where ψ is the energy density (including the gradient terms). Let's suppose we now perform a spatial translation $\delta\mathbf{r}$ on the scalar field $\phi(\mathbf{r})$:

$$\phi(\mathbf{r}) \to \phi(\mathbf{r} - \delta\mathbf{r}) = \phi(\mathbf{r}) \underbrace{-\delta r_\alpha \partial_\alpha \phi}_{\delta\phi}. \tag{2}$$

The energy density must also be translated *via* $\delta\mathbf{r}$, *i.e.*

$$\psi(\mathbf{r}) \to \psi(\mathbf{r} - \delta\mathbf{r}) = \psi(\mathbf{r}) \underbrace{-\delta r_\alpha \partial_\alpha \psi}_{\delta\psi}. \tag{3}$$

Thus the change in the energy density due this spatial translation is:

$$\delta\psi = -\delta r_\alpha \partial_\alpha \psi. \tag{4}$$

However, we also know that:

$$\delta\psi = \left[\frac{\partial\psi}{\partial\phi} - \partial_\alpha\left(\frac{\partial\psi}{\partial(\partial_\alpha\phi)}\right)\right]\delta\phi + \partial_\alpha\left(\frac{\partial\psi}{\partial(\partial_\alpha\phi)}\delta\phi\right). \tag{5}$$

If $\phi(\mathbf{r})$ remains in equilibrium before and after the transformation, $\phi(\mathbf{r})$ must then satisfy Euler-Lagrange equation and the terms inside the square bracket above vanish. Thus we have:

$$\delta\psi = \partial_\alpha\left(\frac{\partial\psi}{\partial(\partial_\alpha\phi)}\delta\phi\right) \quad \text{and} \quad \delta\psi = -\delta r_\alpha \partial_\alpha \psi. \tag{6}$$

In other words,

$$\partial_\alpha\left(\frac{\partial\psi}{\partial(\partial_\alpha\phi)}\delta r_\beta \partial_\beta\phi\right) - \delta r_\beta \partial_\beta \psi = 0 \tag{7}$$

$$\partial_\alpha \underbrace{\left(\frac{\partial\psi}{\partial(\partial_\alpha\phi)}\partial_\beta\phi - \psi\delta_{\alpha\beta}\right)}_{\mathcal{J}_{\alpha\beta}} = 0 \tag{8}$$

$$\mathcal{J}_{\alpha\beta} = \text{constant} \tag{9}$$

We call $\mathcal{J}_{\alpha\beta}$ the Noether current.

For a symmetric Landau energy, we have:

$$\psi = \underbrace{\frac{\alpha}{2}\phi^2 + \frac{\beta}{4}\phi^4}_{f(\phi)} + \frac{\kappa}{2}|\nabla\phi|^2, \tag{10}$$

and the Noether current is:

$$\mathcal{J}_{\alpha\beta} = \kappa(\partial_\alpha\phi)(\partial_\beta\phi) - \left[f(\phi) + \frac{\kappa}{2}|\nabla\phi|^2\right]\delta_{\alpha\beta} = \text{constant} \tag{11}$$

In particular in one-dimension $\alpha = \beta = x$, we get:

$$\mathcal{J}_{xx} = \frac{\kappa}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - f(\phi) = \text{constant}. \tag{12}$$